

# Forecasting economic time series using score-driven dynamic models with mixed-data sampling

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## Abstract

We introduce a mixed-frequency score-driven dynamic model for multiple time series where the score contributions from high-frequency variables are transformed by means of a mixed-data sampling weighting scheme. The resulting dynamic model delivers a flexible and easy-to-implement framework for the forecasting of low-frequency time series variables through the use of timely information from high-frequency variables. We verify in-sample and out-of-sample performances of the model in an empirical study on the forecasting of U.S. headline inflation and GDP growth. In particular, we forecast monthly headline inflation using daily oil prices and quarterly GDP growth using a measure of financial risk. The forecasting results and other findings are promising. Our proposed score-driven dynamic model with mixed-data sampling weighting outperforms competing models in terms of point and density forecasts.

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# 1 Introduction

In many studies concerning the forecasting of economic time series with several variables, we often need to overcome complexities related to the different sampling frequencies at which we observe the variables over time. The challenges of mixed data frequency are reviewed in the context of econometric analysis by [Ghysels and Marcellino \(2016\)](#) and discussed in the context of forecasting by [Armesto et al. \(2010\)](#) and [Andreou et al. \(2011\)](#). In particular, in cases of economic forecasting where both economic and financial variables are relevant, the distinction between low frequency and high frequency data sampling can be substantive. Financial variables, such as stock prices, commodity prices and exchange rates, are typically available at the daily frequency and increasingly at the intraday level (ultra-high frequency) because it is relatively straightforward to electronically record financial transactions. On the other hand, it is more complicated and more costly to collect data on economic variables, such as inflation and gross domestic product (GDP) growth. Hence economic variables are typically available at a quarterly or monthly level. When the interest is in the forecasting of economic variables, the high-frequency financial variables may have a relevant role to play as predictors and may be capable to improve the accuracy of forecasts. Recent researches by [Carriero et al. \(2017\)](#) and [Adrian et al. \(2017\)](#) among others have highlighted the role of variables measuring financial conditions played in forecasting economic variables.

A widely used method for incorporating high frequency data to produce forecasts of low frequency variables is the Mixed Data Sampling (MIDAS) method of [Ghysels et al. \(2004\)](#). MIDAS is a regression-based method that transforms the high frequency variables into low frequency indicators via a parsimonious weighting scheme with possibly different weights to data sampled at high frequency (within the low frequency period). The weighting scheme can reflect the notion that more recent observations should be more informative to predict future values of the low frequency variable. The MIDAS approach (or touch) can be used easily within a (dynamic) regression model but it can also be adopted within other models such as vector autoregressive and dynamic factor models. For instance, [Marcellino and Schumacher \(2010\)](#) have considered a two step approach that combines principal component analysis and MIDAS regressions.

In our current study we adopt a dynamic model with score-driven time-varying loca-

tion and scale parameters. [Creal et al. \(2013\)](#) and [Harvey \(2013\)](#) have developed a general framework to specify time-varying parameter models in an observation-driven setting. The resulting class of models is referred to as Generalized Autoregressive Score (GAS) models. The defining feature of GAS models is that the time-varying parameters are driven by the score of the predictive log-likelihood function. The use of the score as an updating mechanism is intuitive: it can be viewed as a Newton-Raphson update that delivers a better fit, in terms of likelihood maximization, for the next period and conditional on past and current information. The score-driven updates have an optimality property. [Blasques et al. \(2015\)](#) show that the score update is optimal in minimizing the Kullback-Leibler divergence with respect to an unknown true distribution. Score-driven models provide an appealing forecasting method and they have been successfully employed in empirical applications to forecast economic and financial variables, see for instance [Delle Monache and Petrella \(2017\)](#) on forecasting inflation and [Lucas and Zhang \(2016\)](#) and [Blasques et al. \(2016a\)](#) on forecasting exchange rates. In a more general context, the forecasting performance of GAS models is investigated in detail by [Koopman et al. \(2016\)](#). Finally, GAS models are appealing because they are flexible in terms of specification while retaining a simple practical implementation. The estimation of unknown parameters in GAS models can be based on standard likelihood inference that does not require computational-intensive or simulation-based methods.

Our main contribution is the introduction of a flexible and easy-to-implement forecasting method for mixed frequency variables that is based on a score-driven dynamic model. In particular, we consider a factor structure where the score innovations from the high frequency variables are transformed into the low frequency score function via a MIDAS weighting scheme. We name the resulting approach MIDAS-GAS. The MIDAS-GAS model retains all the appealing features of standard GAS models and elevates the MIDAS approach to a more general device for handling mixed frequencies. For example, we illustrate how the MIDAS-GAS framework can be used to specify mixed frequency models with conditional heteroscedastic errors and parameter updates that are robust against outliers. Furthermore, we adopt the weighted likelihood approach of [Blasques et al. \(2016b\)](#) for the likelihood-based estimation of parameters in the MIDAS-GAS model. We discuss how the proposed weighted likelihood method can be reduced to the standard maximum likelihood method when considering only the likelihood contributions of the variable of interest. These devel-

opments deliver a computationally fast and easy-to-implement methodology for parameter estimation, analysis and forecasting. We illustrate the MIDAS-GAS framework to produce forecasts of monthly U.S. headline inflation and GDP growth. In particular, for forecasting monthly inflation we consider daily crude oil inflation as a predictor, and for quarterly GDP we take a daily measure of financial condition as a predictor. Furthermore, we also show how the MIDAS-GAS approach can be used for nowcasting and we apply the method to nowcast GDP growth. We present a detailed account of the forecasting and nowcasting study that includes comparisons with many competing models such as MIDAS regression models, autoregressive models and standard GAS models. The results show a promising performance of the MIDAS-GAS model in terms of point as well as density predictions.

An alternative approach to MIDAS-based methods for the treatment of mixed frequency data is provided by state space time series analyses which rely on the Kalman filter. In this approach we align the data at the highest data sampling frequency and introduce missing observations for the low frequency variables. The Kalman filter is then used to handle these artificial missing observations, see [Mariano and Murasawa \(2003\)](#), [Schumacher and Breitung \(2008\)](#) and [Blasques et al. \(2016b\)](#) for such solutions and with interesting illustrations. A limitation of this more rigorous approach compared to our MIDAS-GAS model is that the Kalman filter requires Gaussian and homoscedastic errors. There is much empirical evidence that shows the importance of accounting for heteroscedastic errors and fat-tailed distributions to obtain more accurate forecasts for economic time series, see, for example, [Creal et al. \(2014\)](#).

We proceed as follows. Section 2 introduces our general modeling framework based on the MIDAS-GAS model and the weighted likelihood method for parameter estimation. Section 3 presents our MIDAS-GAS dynamic factor model with heteroscedastic errors and robust parameter updates. Section 4 illustrates the two empirical applications with the forecasting of monthly U.S. headline inflation and quarterly GDP growth. Section 5 concludes.

## 2 The MIDAS-GAS model

Assume that our aim is to forecast a key economic variable denoted by  $y_t^L$ . The variable is observed sequentially over time at a low data sampling frequency as indicated by  $L$ . We

assume that another related variable  $x_t^H$  can be observed at a high data sampling frequency as indicated by  $H$  where  $L < H$ . This variable is not of interest but we assume that it can be exploited to obtain more accurate forecasts for the key variable  $y_t^L$ . Hence at each time point  $t$  of the low frequency variable  $y_t^L$  we have the predictor  $x_t^H = (x_{1,t}^H, \dots, x_{n_x,t}^H)'$  where  $x_t^H \in \mathbb{R}^{n_x}$  is a vector-valued variable that contains all available high frequency observations within the time period  $t$  and where  $n_x$  is the number of observations of the high frequency variable that is available in time period  $t$ . For example, when we forecast monthly headline inflation, that is  $y_t^L$ , using daily crude oil inflation, that is  $x_t^H$ , we have  $n_x$  equal to the number of working days in a month. For notational convenience and simplicity of exposition, we assume that both variables  $y$  and  $x$  are univariate. However, all results discussed below can be extended to the multivariate case straightforwardly.

## 2.1 The MIDAS touch

Amongst the range of forecasting methods using mixed frequency data, the MIDAS regression is regarded as a simple and direct forecasting method. Denote the  $h$ -step ahead forecast of  $y_T^L$  by  $\hat{y}_{T+h|T}^L$  where  $T$  denotes the sample size. This forecast can be constructed by considering the following  $p$ -lag MIDAS regression, known as multiplicative MIDAS (Chen and Ghysels, 2010; Bai et al., 2013),

$$y_{t+h}^L = c + D_p(B, \beta)y_t^L + D_p(B, \alpha) \sum_{i=1}^{n_x} \omega_i(\varphi)x_{i,t}^H + \epsilon_t, \quad (1)$$

for  $t = 1, \dots, T$ , where  $c$  is an intercept,  $D_p(B, z) = z_0 + z_1B + \dots + z_pB^p$ , for  $z = \beta, \alpha$  is a lag polynomial function with backshift operator  $B$ , unknown parameter vectors  $\beta = (\beta_0, \dots, \beta_p)'$  and  $\alpha = (\alpha_0, \dots, \alpha_p)'$ , and weighting coefficients  $\omega_i(\varphi)$  for  $i = 1, \dots, n_x$  and for a parameter vector  $\varphi$ , and  $\epsilon_t$  is an identically independently distributed (iid) error with mean zero and variance  $\sigma^2$ . Ghysels et al. (2004) advocate a parsimonious weighting function for  $\omega_i(\varphi)$ , for  $i = 1, \dots, n_x$ , based on exponential Almon lag or Beta lag parameterizations. The  $q$ -th order exponential Almon lag is specified as

$$\omega_i(\varphi) = \frac{\exp(\varphi_1 i + \varphi_2 i^2 + \dots + \varphi_q i^q)}{\sum_{i=1}^{n_x} \exp(\varphi_1 i + \varphi_2 i^2 + \dots + \varphi_q i^q)},$$

for some  $q$ -dimensional parameter vector  $\varphi = (\varphi_1, \dots, \varphi_q)'$ . In practice,  $q$  is set equal to 2 which reduces the Almon lag to a normalized exponential quadratic weighting function. The Beta lag is specified as

$$\omega_i(\varphi_1, \varphi_2) = \frac{\text{Beta}(i/n_x; \varphi_1, \varphi_2)}{\sum_{i=1}^{n_x} \text{Beta}(i/n_x; \varphi_1, \varphi_2)},$$

where  $\text{Beta}(\cdot; \varphi_1, \varphi_2)$  is the probability density function of a Beta distribution with parameter vector  $\varphi = (\varphi_1, \varphi_2)'$ . Figure 1 illustrates the weighting functions based on second order exponential Almon lag and Beta lag; we can conclude that the shapes of these weighting functions are very flexible. The parameters of the MIDAS regression include  $c$ ,  $\beta$ ,  $\alpha$ ,  $\varphi$  and  $\sigma^2$  which can be estimated using either nonlinear least squares or maximum likelihood (ML). We obtain the  $h$ -step ahead forecast  $\hat{y}_{T+h|T}^L$  in the usual way. Andreou et al. (2011), and the references therein, present a more detailed discussion on MIDAS regressions.

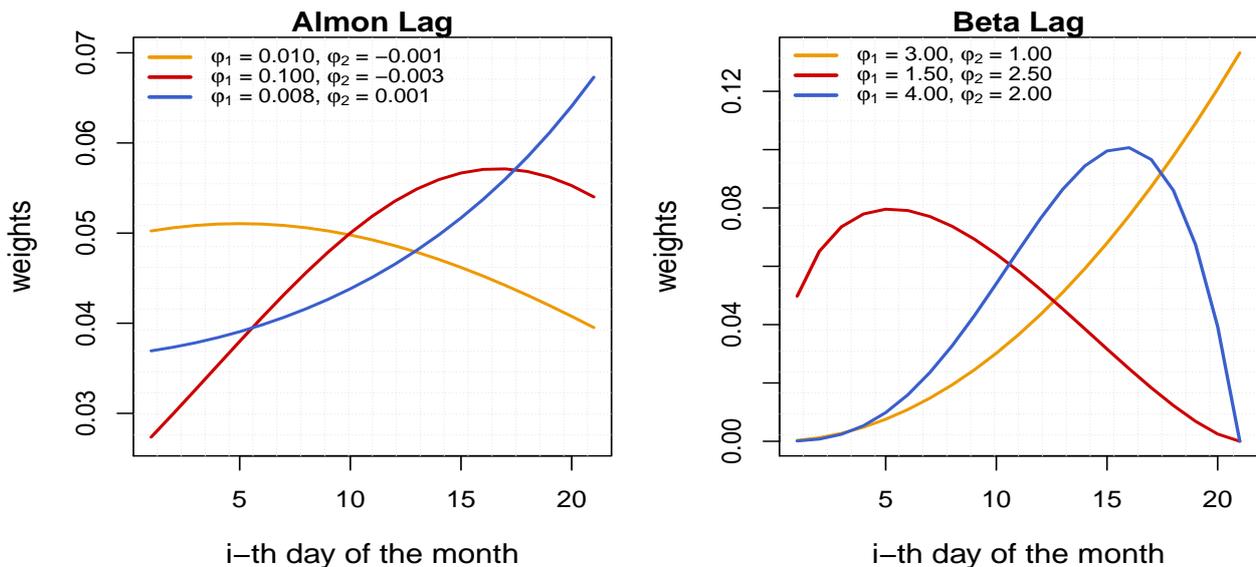


Figure 1: Weighting functions based on exponential Almon lag and Beta lag. The weights are assigned to daily observations within a month using different parameters of the exponential Almon lag (left) and the Beta lag (right) functions.

## 2.2 The MIDAS-GAS filter

We incorporate the mixed data sampling method using a parsimonious weighting function for the class of score-driven models which are proposed by Creal et al. (2013) and Harvey

(2013). We refer to the resulting framework as MIDAS-GAS. The GAS filter provides a convenient way of modeling time-varying parameters in an econometric model. Time-varying parameters are specified as autoregressive processes where the innovations are the scores of the predictive log-likelihood function. This approach delivers an observation-driven specification that facilitates inference because the likelihood function is available in closed form through prediction error decomposition. The derivation of the MIDAS-GAS filter is given below.

We consider a multivariate conditional distribution for the observable variables  $y_t^L$  and  $x_t^H$  of the form

$$y_t^L, x_t^H | f_t \sim p(y_t^L, x_t^H | f_t; \psi), \quad (2)$$

where  $p(\cdot | f_t; \psi)$  is a  $(n_x + 1)$ -variate conditional density,  $f_t$  is a time-varying parameter and  $\psi$  is a vector of static parameters. Further we assume that  $y_t^L$  is independent of  $x_t^H$  conditional on  $f_t$  and also that the elements of the vector  $x_t^H$  are iid conditional on  $f_t$ . These assumptions are quite standard in the literature of multivariate GAS models and also state-space models in general. Note also that the conditional independence does not imply that the variables are independent because the dependence is determined by the common time-varying parameter  $f_t$ . Under these conditions, the joint conditional density of  $y_t^L$  and  $x_t^H$  can be factorized as follows

$$p(y_t^L, x_t^H | f_t; \psi) = p_y(y_t^L | f_t; \psi) \prod_{i=1}^{n_x} p_x(x_{i,t}^H | f_t; \psi), \quad (3)$$

where  $p_y(\cdot | f_t; \psi)$  is the conditional density function of  $y_t^L$  and  $p_x(\cdot | f_t; \psi)$  is the conditional density function of  $x_{i,t}^H$ .

In the standard GAS framework, the time varying parameter  $f_t$  is specified as an autoregressive process driven by the score of the predictive log-density in (3). Under some standard differentiability conditions, the score  $\nabla_t = \partial \log p(y_t^L, x_t^H | f_t; \psi) / \partial f_t$  is given by

$$\nabla_t = \nabla_t^y + \sum_{i=1}^{n_x} \nabla_{i,t}^x,$$

where  $\nabla_t^y = \partial \log p_y(y_t^L | f_t; \psi) / \partial f_t$  and  $\nabla_{i,t}^x = \partial \log p_x(x_{i,t}^H | f_t; \psi) / \partial f_t$ . In the GAS literature, the score in  $\nabla_t$  is sometimes rescaled to account for the curvature of the likelihood, see [Creal et al. \(2013\)](#) for a more detailed discussion. By allowing some rescaling of the score, we

define the score innovation as

$$s_t = s_t^y + \sum_{i=1}^{n_x} s_{i,t}^x,$$

where  $s_t^y = S_t^y \nabla_t^y$  and  $s_{i,t}^x = S_{i,t}^x \nabla_t^{x_i}$  for some given scaling factors  $S_t^y$  and  $S_{i,t}^x$ . For instance, these scaling factors can be chosen to be some transformation of the Fisher information or they can be simply set equal to one. The score innovation  $s_t$  is easy to interpret:  $s_t^y$  can be seen as the information from  $y_t^L$  to update the time-varying parameter  $f_t$ , similarly,  $s_{i,t}^x$  can be seen as the information from  $x_{i,t}^H$  to update  $f_t$ . In standard GAS models,  $s_t$  is directly used as innovation for the time varying parameter  $f_t$ . This means that the source of information are equally weighted over the score contributions  $s_{i,t}^x$ ,  $i = 1, \dots, n_x$ . Since our objective is forecasting the variable  $y_t^L$ , it is quite restrictive to assume that the predictive content carried by the elements  $s_{i,t}^x$ ,  $i = 1, \dots, n_x$ , is the same. For instance, in some situations it may be reasonable to think that the latest score innovations within the time period  $t$  may be more informative in predicting  $y_{t+1}^L$ , given that they are closer in time. We therefore introduce a MIDAS weighting scheme for the score innovations  $s_{i,t}^x$ , that is,  $\sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^x$ . In this way, the more recent score innovations are allowed to receive more weight. The resulting MIDAS-GAS filter takes the simple form

$$f_{t+1} = \delta + \beta f_t + \alpha_y s_t^y + \alpha_x \sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^x, \quad (4)$$

where  $\delta$ ,  $\beta$ ,  $\alpha_y$ ,  $\alpha_x$  and  $\varphi$  are static parameters to be estimated. The parameter  $\beta$  is the autoregressive coefficient of  $f_t$  and the parameters  $\alpha_y$  and  $\alpha_x$  determine the relative importance of  $y_t^L$  and  $x_t^H$  in predicting future values of  $y_t^L$ . Notice that the autoregressive dynamics of  $f_t$  in (4) is of order one, MIDAS-GAS(1,1). However, it is straightforward to extend equation (4) to higher orders by adding lags of  $f_t$  and lags of the score-innovations. Furthermore, in general,  $f_t$  can be a vector of time-varying parameters.

The MIDAS-GAS model specified by equations (2)-(4) is very general: it allows a wide class of observation densities to be considered. For instance, the MIDAS-GAS filter can be employed when the observed variables  $y_t^L$  and  $x_t^H$  are ordinal or categorical and densities such as the ordered logit are considered. We refer the reader to [Creal et al. \(2014\)](#) for possible applications in this context. We present some MIDAS-GAS specifications with

dynamic mean and variance in Section 3. These specifications are well suited to obtain point and density forecasts of economic variables. Additionally, in Section 3, we show that the MIDAS-GAS approach nests the multiplicative MIDAS model in (1) when a Gaussian distribution for the error term is considered.

### 2.3 Weighted likelihood estimation

One of the appealing features of GAS models is that the likelihood function is available in closed form through prediction error decomposition. Therefore ML estimation is easy to implement and computationally fast. This is also the case for our MIDAS-GAS model in (2)-(4). For our MIDAS-GAS model, we need to estimate the parameter vector  $\theta = (\psi', \delta, \beta, \alpha_y, \alpha_x, \varphi')$  that contains all the static parameters of the model. In the following, instead of ML, we consider a weighted maximum likelihood (WML) approach as proposed by Blasques et al. (2016b). Blasques et al. (2016b) show that in a Gaussian dynamic factor model with mixed frequency data, forecasting can be improved using parameters estimated via a WML procedure. Its novelty is to introduce variable-specific weights instead of observation-specific weights in the likelihood function. In our setting, this method is particularly appealing because we are only interested in forecasting the low frequency variable  $y_t^L$  and the high frequency variable  $x_t^H$  is just used as a predictor. Following Blasques et al. (2016b), the weighted likelihood of the MIDAS-GAS model can be written as

$$L_T^W(\theta) = \sum_{t=1}^T \log p_y(y_t^L | f_t; \psi) + W \sum_{t=1}^T \sum_{i=1}^{n_x} \log p_x(x_{i,t}^H | f_t; \psi),$$

for a predetermined weight  $W \in [0, 1]$ . If the weight  $W$  is equal to one, then the weighted likelihood function is equal to the usual likelihood function, instead, if  $W$  is equal to zero, then only the likelihood of the variable  $y_t^L$  is considered. We highlight that setting  $W = 0$  may lead to lack of identifiability of some parameters and this needs to be accounted when dealing with specific models. Some parameter restrictions to ensure identifiability are discussed in Section 4. Maximization of the weighted likelihood function can be done via standard numerical routines. In general, the weight  $W$  cannot be estimated together with the other parameters, so cross validation approach can be used to select the weight. In the empirical

application of Section 4, we set the weight to zero and therefore consider only the likelihood contribution of the univariate time series  $y_t^L$ . As shown in Blasques et al. (2016b), in principle, a different choice of the weight may provide better out-of-sample results. However, we consider  $W = 0$  to obtain a simple form of the likelihood function and to have a more realistic forecasting comparison with other models. In case we set  $W$  equal to zero, the MIDAS-GAS model can be regarded as a univariate model for  $y_t^L$  of the form

$$y_t^L \sim p_y(y_t^L | f_t; \psi), \quad f_{t+1} = \delta + \beta f_t + \alpha_y s_t^y + \alpha_x \sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^x.$$

Although the weight  $W$  is set to zero, the high frequency variable  $x_t^H$  still enters into the MIDAS-GAS time-varying parameter  $f_t$  through the score innovations  $s_{i,t}^x$ . This model can be viewed as a univariate GAS model with an exogenous predictor  $x_t^H$ . Therefore, standard asymptotic theory for ML estimation of GAS models in Blasques et al. (2014) applies in this case.

### 3 A MIDAS-GAS factor model

In this section, we develop some factor models for mixed frequency data based on the general MIDAS-GAS specification in (2)-(4). We can formulate a factor forecasting model, which is essentially a multivariate time-varying location and scale model, with mixed frequency data as follows

$$\begin{bmatrix} y_t^L \\ x_t^H \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_\mu \mathbf{1}_{n_x} \end{bmatrix} \mu_t + \sigma_t \begin{bmatrix} \epsilon_t^y \\ \epsilon_t^x \end{bmatrix}, \quad (5)$$

where  $\epsilon_t^y$  and  $\epsilon_t^x = (\epsilon_{1,t}^x, \dots, \epsilon_{n_x,t}^x)'$  are independent disturbance terms that follow some parametric distributions. The error  $\epsilon_t^y$  has mean zero and unit variance, instead,  $\epsilon_{i,t}^x$  has mean zero and variance  $\lambda_\sigma$ . The vector  $\mathbf{1}_{n_x}$  is a vector of length  $n_x$  that contains ones, and  $\lambda_\mu$  is a loading on the factor  $\mu_t$  for  $x_t^H$ . The specification in (5) entails a factor structure for both  $\mu_t$  and  $\sigma_t^2$  where  $\lambda_\sigma$  and  $\lambda_\mu$  are the corresponding loadings for the high frequency variable. The inclusion of a dynamic scale  $\sigma_t^2$  enables the model to capture conditional heteroscedasticity in the data. This can be particularly useful to improve density forecasts.

We specify the dynamic location  $\mu_t$  and scale  $\sigma_t^2$  according to the MIDAS-GAS updating

equation in (4), that is,

$$\begin{aligned}\mu_{t+1} &= \delta_\mu + \beta_\mu \mu_t + \alpha_\mu^y s_t^y + \alpha_\mu^x \sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^x, \\ \sigma_{t+1}^2 &= \delta_\sigma + \beta_\sigma \sigma_t^2 + \alpha_\sigma^y v_t^y + \alpha_\sigma^x \sum_{i=1}^{n_x} \omega_i(\varphi) v_{i,t}^x,\end{aligned}\tag{6}$$

where  $s_t^y$  and  $s_{i,t}^x$  are the score contributions for the time varying mean  $\mu_t$  and from the variables  $y_t^L$  and  $x_{i,t}^H$ , respectively, and  $v_t^y$  and  $v_{i,t}^x$  are the score contributions for the time varying variance  $\sigma_t^2$  and from  $y_t^L$  and  $x_{i,t}^H$ , respectively. The functional forms of the score innovations  $s_t^y$ ,  $s_{i,t}^x$ ,  $v_t^y$  and  $v_{i,t}^x$  depend on the choice of the distributions of the error terms  $\epsilon_t^y$  and  $\epsilon_t^x$ . In the following we consider two different specifications for the error terms: the Normal and the Student-t distribution. The use of a Student-t distribution leads to robust updating of the time varying mean and variance.

In the first specification, we consider the errors to be normally distributed, i.e.  $\epsilon_t^y \sim N(0, 1)$  and  $\epsilon_{i,t}^x \sim N(0, \lambda_\sigma)$ . Under this specification and considering a scaling factor for the scores based on the Fisher information, we obtain that, up to a constant scale, the score innovations are given by

$$\begin{aligned}s_t^y &= y_t^L - \mu_t, & s_{i,t}^x &= x_{i,t}^H - \lambda_\mu \mu_t, \\ v_t^y &= (y_t^L - \mu_t)^2 - \sigma_t^2, & v_{i,t}^x &= (x_{i,t}^H - \lambda_\mu \mu_t)^2 - \lambda_\sigma \sigma_t^2.\end{aligned}\tag{7}$$

We note that the specification in (6) and (7) nests the multiplicative MIDAS model in (1). In particular, by setting  $\beta_\mu = \alpha_\mu^y + \alpha_\mu^x \lambda_\mu$  and  $\alpha_\sigma^y = \alpha_\sigma^x = \beta_\sigma = 0$ , the model for  $y_t$  becomes

$$y_t = \mu_t + \sqrt{\delta_\sigma} \epsilon_t^y, \quad \mu_t = \delta_\mu + \alpha_\mu^y y_{t-1} + \alpha_\mu^x \sum_{i=1}^{n_x} \omega_i(\varphi) x_{i,t-1}^H.$$

This model indeed corresponds to the multiplicative MIDAS model of order one. As mentioned before, the MIDAS-GAS can be extended to higher orders by including more lags of  $\mu_t$  and of the score innovations. Therefore, in a similar way the factor MIDAS-GAS can nest multiplicative MIDAS of any order when a Gaussian distribution for the error term is considered.

In the second specification, we model  $\epsilon_t^y$  considering a Student-t distribution with zero mean and unit variance, i.e.  $\epsilon_t^y \sim t_\nu(0, 1)$ . In this way, the conditional density function of  $y_t^L$  is given by

$$p_y(y_t^L | \mu_t, \sigma_t^2; \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)\sigma_t^2}} \left(1 + \frac{(y_t - \mu_t)^2}{(\nu-2)\sigma_t^2}\right)^{-\frac{\nu+1}{2}},$$

where  $\nu > 2$  represents the degrees of freedom of the Student-t error  $\epsilon_t^y$ . The use of a Student-t distribution can be particularly important for density forecasts when the time series of interest  $y_t^H$  exhibits fat tails. Furthermore, as we shall see, the Student-t delivers a robust update for our MIDAS-GAS model and this can also lead to an improvement in point forecasts. Note that a Student-t distribution for the error term  $\epsilon_t^x$  can also be considered. In the following, for simplicity, we consider a normal distribution for  $\epsilon_t^x$ . However, we highlight that the choice of the distribution of  $\epsilon_t^x$  is less relevant than the one of  $\epsilon_t^y$  because we are not interested in forecasting  $x_t^H$  and also the estimation of the model is based on WML with  $W = 0$ . Under this second specification, we obtain that, up to some scaling constant, the score innovations have the following form

$$\begin{aligned} s_t^y &= \frac{(\nu+1)(y_t^L - \mu_t)}{(\nu-2) + (y_t^L - \mu_t)^2 \sigma_t^{-2}}, & s_{i,t}^x &= x_{i,t}^H - \lambda_\mu \mu_t, \\ v_t^y &= \frac{(\nu+1)(y_t^L - \mu_t)^2}{(\nu-2) + (y_t^L - \mu_t)^2 \sigma_t^{-2}} - \sigma_t^2, & v_{i,t}^x &= (x_{i,t}^H - \lambda_\mu \mu_t)^2 - \lambda_\sigma \sigma_t^2. \end{aligned} \quad (8)$$

The score innovations  $s_t^y$  and  $v_t^y$  in (8) are robust against outliers. This is one of the peculiarities of the GAS approach when using fat-tailed distributions, see [Harvey and Luati \(2014\)](#) and [Harvey \(2013\)](#) for a discussion on robust score updates with the Student-t distribution.

We can identify three differences between the MIDAS-GAS model and the widely used Gaussian linear dynamic factor model. First, we do not restrict  $\epsilon_t^y$  and  $\epsilon_t^x$  to come from Gaussian distributions. Secondly, the dynamic factor process is non-linear due to the score updating mechanism, which locally minimizes the KL divergence between the true measure of the data and the one implied by the factor model as discussed in [Blasques et al. \(2015\)](#). Third, the MIDAS touch incorporated in the dynamic factor process (6) further balances the predictive information from  $y_t^L$  and  $x_t^H$ . This differs from the MIDAS factor model of [Marcellino and Schumacher \(2010\)](#), which extract factors from  $x_t^H$  and treat the high frequency factor as regressors in the standard MIDAS regression. It is also different from

the MIDAS dynamic factor model of [Frailé and Monteforte \(2011\)](#), which is basically a bivariate Gaussian model with observation vector  $(y_t^L, \sum_{i=1}^{n_x} \omega_i(\varphi)x_{i,t}^H)'$  and a single factor. Both models require Gaussian errors and a linear structure so that principal component analysis (PCA) and Kalman filter can be used to extract factors and produce forecasts. The GAS filter in our model has the appealing feature that it does not limit us in a Gaussian and linear world. As a result, density forecasts are improved by incorporating stochastic volatility and fat-tailed errors. This has been extensively documented by the rich literature studying the forecasting performance of models with stochastic volatility, for example [Kim et al. \(1998\)](#), [Tse and Tsui \(2002\)](#), and [Chib et al. \(2002\)](#) among others. However, estimation of such models is computationally demanding and Bayesian methods are often used. Instead, the estimation of MIDAS-GAS models is straightforward.

## 4 Empirical applications

In this section, we employ the MIDAS-GAS factor model for forecasting U.S. headline inflation and GDP growth. In the first application, we forecast monthly headline inflation using daily crude oil inflation. In the second application, we forecast quarterly GDP growth using a daily measure of financial condition obtained from the S&P 500 stock index. In this way, we can assess the performance of the MIDAS-GAS model with different frequencies and different variables.

### 4.1 Forecasting monthly inflation with daily oil prices

#### 4.1.1 The dataset and in-sample results

There is evidence in the literature that oil prices have relevant predictive content for U.S. inflation, see [Clark and Terry \(2010\)](#) and [Stock and Watson \(2003\)](#) for instance. We consider time series of monthly U.S. headline inflation and crude oil inflation from January 1986 to August 2018. [Figure 2](#) shows the corresponding time series plots.

We consider the MIDAS-GAS factor model presented in [Section 3](#) based on WML with  $W = 0$ . We impose some parameter restrictions. In particular, we consider a random walk process for the time varying mean by imposing  $\beta_\mu = 1$  and  $\delta_\mu = 0$ . Furthermore, we set

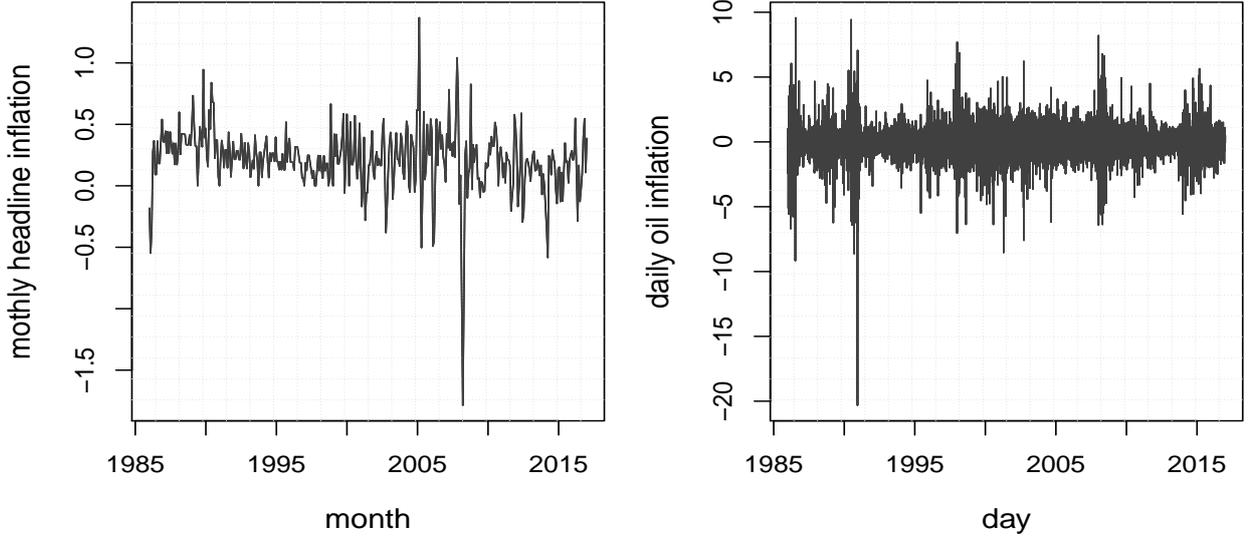


Figure 2: The headline CPI inflation and oil price inflation. Left: The monthly headline inflation computed as the first difference of the logarithm of monthly CPI indices. Right: The daily oil price inflation computed as the first difference of the logarithm of daily WTI crude oil prices.

$\lambda_\sigma = 1$  to ensure identifiability. Note that  $\lambda_\sigma$  could be set equal to any positive value and the resulting model would be equivalent, up to a reparameterization. Finally, we consider a second order exponential Almon lag function for the MIDAS weights with  $\varphi = (\varphi_1, \varphi_2)'$ . The factor MIDAS-GAS model with the above restrictions can be written as a univariate model for  $y_t^L$  of the form

$$\begin{aligned}
 y_t^L &= \mu_t + \sigma_t \epsilon_t^y, & \mu_{t+1} &= \mu_t + \alpha_\mu^y s_t^y + \alpha_\mu^x \sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^x, \\
 \sigma_{t+1}^2 &= \delta_\sigma + \beta_\sigma \sigma_t^2 + \alpha_\sigma^y v_t^y + \alpha_\sigma^x \sum_{i=1}^{n_x} \omega_i(\varphi) v_{i,t}^x, & & (9)
 \end{aligned}$$

where the score innovations have the functional form either as in (7) or (8), depending on the choice of the error distribution.

We estimate 4 different specifications of the MIDAS-GAS model in (9). The t-MIDAS-GASg has a Student-t error for  $y_t^L$  and the score innovations given in (8), the t-MIDAS-GAS is the same as the t-MIDAS-GASg but without conditional heteroscedasticity, i.e.  $\beta_\sigma = \alpha_\sigma^x = \alpha_\sigma^y = 0$ , the MIDAS-GASg has a normal error for  $y_t^L$  and the score innovations given in (7), and the MIDAS-GAS is the same as the MIDAS-GASg but with constant variance, i.e.  $\beta_\sigma = \alpha_\sigma^x = \alpha_\sigma^y = 0$ . Table 1 reports the estimation results. We can see that the Student-t

	$\nu$	$\lambda_{\mu}^x$	$\alpha_{\mu}^y$	$\alpha_{\mu}^x$	$\alpha_{\sigma}^y$	$\alpha_{\sigma}^x$	$\delta_{\sigma}$	$\beta_{\sigma}$	lik	AIC
t-MIDAS-GASg	4.86	0.13	0.14	0.11	0.34	0.01	3.48	0.83	-892.81	<b>4.61</b>
t-MIDAS-GAS	4.24	0.16	0.30	0.18	-	-	7.62	-	-926.23	4.76
MIDAS-GASg	-	0.14	0.18	0.10	0.24	0.00	4.63	0.91	-906.13	4.67
MIDAS-GAS	-	0.15	0.28	0.15	-	-	8.57	-	-977.24	5.02

Table 1: Full-Sample parameter estimates of MIDAS-GAS factor models using monthly inflation with daily oil prices. The last two columns report the log-likelihood and the average AIC criterion, respectively.

distribution improves the in-sample fit. This can be noted from the better performance of the models with Student-t error in terms of Akaike information criterion (AIC). Additionally, the estimated degrees of freedom parameters  $\nu$  are very small (around 4), indicating the presence of fat tails. The better fit of the Student-t is not surprising since the headline inflation series exhibits several extreme observations, see Figure 2. Furthermore, we also note that the conditional heteroscedasticity of the error term delivers a clear in-sample improvement in terms of AIC. Also this finding is coherent with the volatility clustering of the inflation series that we can see in Figure 2.

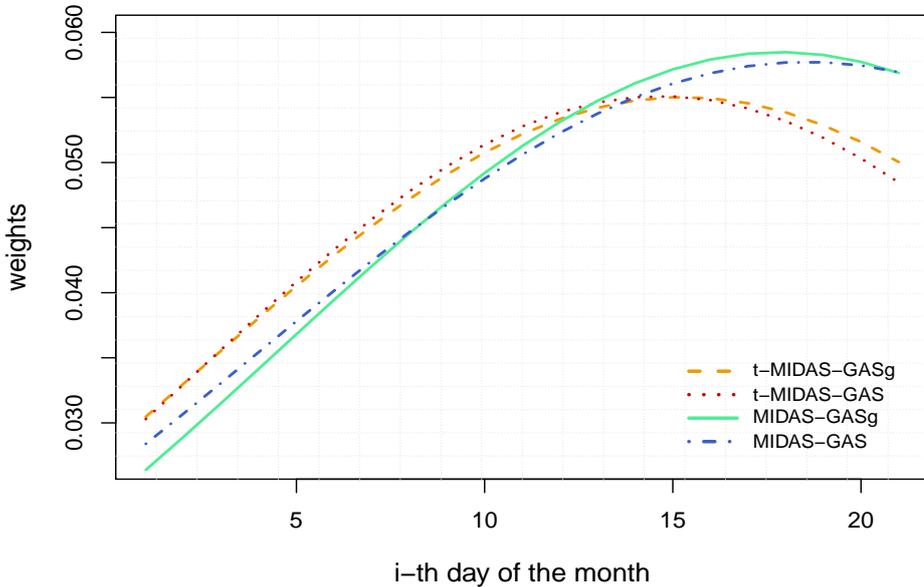


Figure 3: Estimated MIDAS-GAS weighting functions for the four model specifications using monthly inflation with daily oil prices. The horizontal axis indicates the day in the month (there are about 21 working days in a month). More weight on the last days of the month indicates that more recent observations are more relevant.

Finally, Figure 3 reports the estimated MIDAS weighting functions for the four different

model specifications. We can see that estimated functions give more weight to the more recent observations of the high frequency variable. This result is coherent with the idea that observations that are farther away in time should be less relevant for forecasting the future values of the variable of interest. Furthermore, we also note that the results seem consistent across the different models since the shape of the four weighting functions is very similar.

#### 4.1.2 Out-of-sample exercise

We consider two forecasting tasks: point forecast and density forecast. Point forecast is a core task carried out in central banks. For example, inflation forecast facilitates the use of forward-looking monetary policy which helps calculate the ex ante real interest rate to determine the aggregate demand or IS curve for an economy. Additionally, density forecast is important because it provides a risk metric measuring how accurate the point forecast is. We evaluate the performance of point forecasts using the Forecast Mean Squared Error (FMSE) and density forecasts using the log score criterion. The log score criterion is a standard method for evaluating density forecasts that is based on Kullback-Leibler divergence, see for instance [Geweke and Amisano \(2011\)](#).

We split the full data sample, which consists of 392 months from April 1986 to August 2018, into two subsamples: the first 187 months are used as in-sample training period and the remaining 200 months are used for the out-of-sample evaluation period. We consider a rolling window forecasting exercise. Therefore, the length of the in-sample estimation period is equal to 187 for all forecasts. We consider multi-step forecasts: from 1 month ahead to 6 months ahead. We highlight that the forecasts have the same horizon for the high and low frequency variables. For instance, the 1 month ahead forecast of January CPI is obtained using data up to the end of December only. Besides our 4 MIDAS-GAS models, we include several competing models for comparison. We include MIDAS regression models, autoregressive models, standard GAS models and the MIDAS factor model of [Franses and Monteforte \(2011\)](#). For these models, we consider Student-t error distributions and conditional heteroscedasticity. [Table 2](#) illustrates the specification of the competing models included in the comparison. The MIDAS regression models and the autoregressive models are estimated as a direct forecasting method for each forecasting horizon. The statistical significance of the difference in performance of each model, compared to a benchmark model

(t-MIDAS-GASg), is tested using the Diebold-Mariano (DM) test (Diebold and Mariano, 1995). We note that, in general, the DM test is problematic when testing nested models. However, the DM test remains valid when a rolling window approach is considered (Giacomini and White, 2006).

	Model description
t-MIDASg( $p$ )	The $p$ -lag MIDAS regression in (1) with Student-t and conditional heteroscedastic error.
t-MIDAS( $p$ )	The $p$ -lag MIDAS regression in (1) with Student-t error.
MIDASg( $p$ )	The $p$ -lag MIDAS regression in (1) with normal and conditional heteroscedastic error.
MIDAS( $p$ )	The $p$ -lag MIDAS regression in (1) with normal error.
t-ARg( $p$ )	Autoregressive model of order $p$ with Student-t and conditional heteroscedastic error.
t-AR( $p$ )	Autoregressive model of order $p$ with Student-t error
ARg( $p$ )	Autoregressive model of order $p$ with normal and conditional heteroscedastic error.
AR( $p$ )	Autoregressive model of order $p$ with normal error.
t-GASg	Standard GAS model with Student-t and conditional heteroscedastic error.
t-GAS	Standard GAS model with Student-t error.
GASg	Standard GAS model with normal and conditional heteroscedastic error.
GAS	Standard GAS model with normal error.
fMIDAS	The MIDAS factor model of Frale and Monteforte (2011).

Table 2: Specification of the competing models used in the out-of-sample exercise.

Table 3 reports the results of the forecasting study. We can see that MIDAS-GAS models tend to have the best performance in terms of point forecasts, except for 1 step ahead forecasts where variants and extensions of multiplicative MIDAS regressions perform equally well. Furthermore, we note that the inclusion of conditional heteroscedasticity and Student-t errors also plays a major role. This can be noted from the fact that the t-MIDAS-GASg model tends to have the best performance among the MIDAS-GAS models. We obtain a similar result for density forecasts. Here the t-MIDAS-GASg has the best performance for several forecasting horizons. Overall we can conclude that MIDAS-GAS models deliver accurate forecasts compared to a wide pool of competing models.

## 4.2 Forecasting GDP growth rate with volatility measures

### 4.2.1 The dataset and in-sample results

In the following, we report the results of a second empirical study where we employ the MIDAS-GAS model for forecasting quarterly U.S. GDP growth using a simple daily measure

	Forecast Mean Squared Error						Log score criterion					
	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
t-MIDAS-GASg	1.00	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>-2.45</b>	<b>-2.63</b>	<b>-2.65</b>	<b>-2.63</b>	-2.62	-2.63
t-MIDAS-GAS	0.99	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	-2.48	-2.68	-2.68	<b>-2.63</b>	<b>-2.61</b>	<b>-2.61</b>
MIDAS-GASg	0.99	1.01	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	-2.51**	-2.80**	-2.86*	-2.87*	-2.90**	-2.96*
MIDAS-GAS	0.99	1.01	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	-2.68*	-2.97*	-2.97*	-2.94	-2.94	-2.96
t-MIDASg(2)	0.98	1.02	1.03*	1.03*	1.02**	1.02*	-2.46	-2.67	-2.74**	-2.73***	-2.71***	-2.71***
t-MIDAS(2)	0.97	1.05	1.03	1.05**	1.03**	1.03*	-2.57	-2.78*	-2.75	-2.75*	-2.72	-2.71
MIDASg(2)	<b>0.95</b>	1.01	1.02*	1.03*	1.02	1.03*	-2.52	-2.84**	-2.93**	-2.99***	-3.00**	-3.13*
MIDAS(2)	<b>0.95</b>	1.02*	1.03*	1.06*	1.04**	1.04*	-2.79*	-3.07*	-3.08*	-3.09*	-3.05*	-3.09*
t-MIDASg(4)	1.00	1.04	1.05*	1.06**	1.04**	1.03**	-2.49	-2.69	-2.74**	-2.72***	-2.72***	-2.72***
t-MIDAS(4)	1.01	1.03*	1.03**	1.05**	1.04***	1.03***	-2.61**	-2.80*	-2.79*	-2.79*	-2.79**	-2.80*
MIDASg(4)	1.00	1.05*	1.07*	1.07**	1.05**	1.03**	-2.55	-2.80*	-2.90**	-2.87**	-2.89**	-2.93*
MIDAS(4)	1.00	1.05*	1.06*	1.08**	1.06**	1.03*	-2.79*	-3.09*	-3.14*	-3.12*	-3.08*	-3.07*
t-ARg(2)	1.07**	1.03	1.01	1.01***	1.01***	1.01**	-2.56***	-2.67	-2.67	-2.66*	-2.66*	-2.65
t-AR(2)	1.08**	1.04	1.02	1.03**	1.02**	1.01**	-2.59**	-2.70	2.64	-2.64	-2.63	-2.62
ARg(2)	1.07**	1.01	<b>1.00</b>	1.01*	1.01*	1.01	-2.63**	-2.83*	-2.91*	-2.92**	-2.96**	-3.02*
AR(2)	1.06	1.03	1.02	1.03**	1.02**	1.01**	-2.82*	-3.03*	-3.00	-2.98*	-2.98*	-2.97*
t-ARg(4)	1.08**	1.04	1.02*	1.01*	1.01	1.01*	-2.58***	-2.69*	-2.69*	-2.66*	-2.66*	-2.65
t-AR(4)	1.08**	1.05	1.02	1.03*	1.02*	1.01**	-2.62***	-2.70	-2.66	-2.64	-2.62	-2.62
ARg(4)	1.09***	1.04*	1.02*	1.02	1.01*	1.01*	-2.65***	-2.85**	-2.93**	-2.92**	-2.95**	-3.02*
AR(4)	1.08*	1.06*	1.04*	1.04**	1.02**	1.01**	-2.86**	-3.13*	-3.12*	-3.08*	-3.04*	-3.03
t-GASg	1.11***	1.05*	1.05**	1.03	1.04	1.02	-2.59***	-2.74**	-2.77***	-2.72*	-2.67	-2.72**
t-GAS	1.15***	1.03	1.04	1.04	1.03	1.02	-2.68***	-2.72*	-2.70	-2.66	-2.63	-2.64
GASg	1.08***	1.02	1.01*	1.01*	1.01*	1.01	-2.63***	-2.86*	-2.95*	-2.94*	-2.99**	-3.05*
GAS	1.06*	1.01	1.01*	1.01*	1.01*	1.01*	-2.83**	-3.03*	-3.05*	-3.00	-3.00	-3.02
fMIDAS	1.13***	1.03	1.03	1.03	1.03	1.02	-2.88**	-2.97*	-2.97	-2.94	-2.83	-2.91*

Table 3: Relative MSE and log score criterion for the different model specifications using monthly inflation with daily oil prices. The first 6 columns of the table report the ratio between the MSE of each model with respect to the MSE of benchmark model (t-MIDAS-GASg) for several forecasting horizons (from 1 to 6 steps ahead). A value greater than 1 indicates that a model is underperforming the benchmark model instead the opposite is true when the MSE ratio is smaller than 1. The last 6 columns report the log score criterion for several forecasting horizons (from 1 to 6 steps ahead). Stars indicate the significance level of the DM test (\*\*\*)0.1%, \*\*1% and \*5%). Bold characters indicate the best performing model for a given horizon.

of financial condition. In particular, we consider the log-squares of the S&P 500 log-returns as predictor; i.e.  $x_{it} = -\log(r_{it}^2)$ , where  $r_{it}$  denotes the log returns of the S&P 500 index on day  $i$  in month  $t$ . The use of this type of financial risk measure to forecast GDP has gained increasing popularity. Orlik and Veldkamp (2014) and Johannes et al. (2016) show that agents form beliefs on economic growth using signals of financial risk. Furthermore, much recent literature has found GDP vulnerability and downside risk of economic growth are preceded by volatility increase from low levels (see, e.g. Gourio, 2012; D’Agostino et al., 2013). These findings are also in line with recent literature on intermediary asset pricing (Brunnermeier and Sannikov, 2014). We use this simple indicator of (the inverse of) financial volatility to illustrate the implementation of MIDAS-GAS models because it is available on a daily basis with long enough historical data. More comprehensive financial condition

indicators such as the weekly NFCI (The National Financial Conditions Index) constructed from 105 measures of financial risks including equity volatility, credit spreads and the term spread by the Federal Reserve Bank of Chicago are available but with much shorter history.

The time series we consider are from the first quarter of 1950 to the second quarter of 2018. Figure 4 shows the quarterly GDP growth series and the daily measure of financial risk.

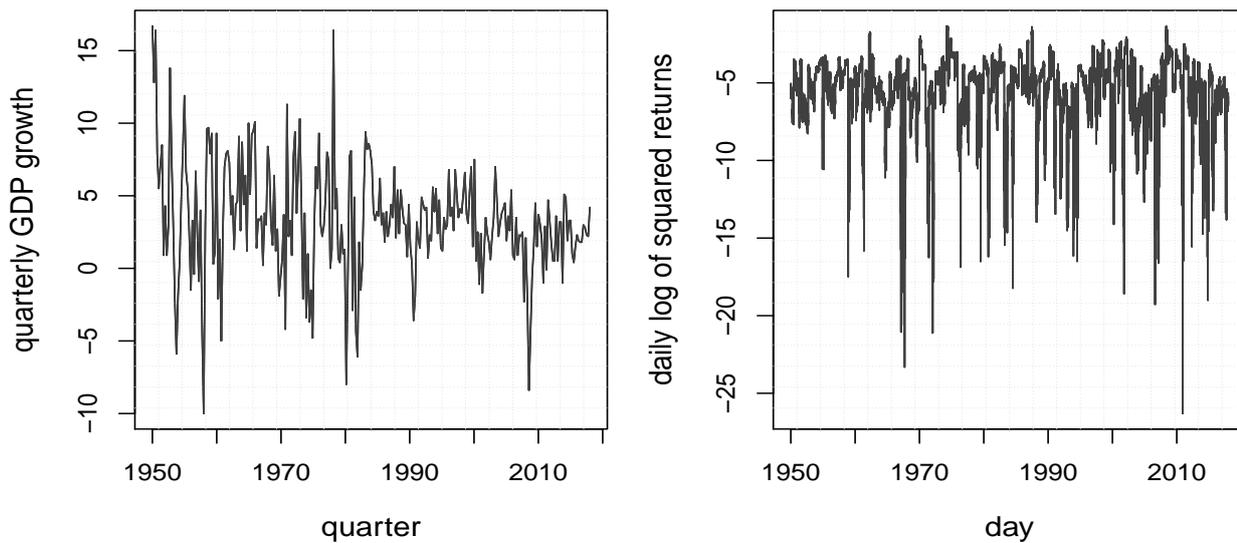


Figure 4: The GDP growth rate and the financial risk measure. Left: The quarterly GDP growth rate. Right: The daily financial risk measure,  $x_{i,t} = -\log(r_{i,t}^2)$ .

	$\nu$	$\lambda_{\mu}^x$	$\alpha_{\mu}^y$	$\alpha_{\mu}^x$	$\alpha_{\sigma}^y$	$\alpha_{\sigma}^x$	$\delta_{\sigma}$	$\beta_{\sigma}$	llik	AIC
t-MIDAS-GASg	9.12	1.94	0.24	0.17	0.13	0.00	9.11	0.98	-700.51	<b>5.19</b>
t-MIDAS-GAS	7.12	1.79	0.28	0.21	-	-	11.66	-	-719.81	5.31
MIDAS-GASg	-	1.86	0.33	0.18	0.13	0.00	12.06	0.97	-704.68	5.21
MIDAS-GAS	-	1.81	0.32	0.21	-	-	11.75	-	-726.36	5.35

Table 4: Full-Sample parameter estimates of MIDAS-GAS factor models using the quarterly GDP growth rate and the daily measure of financial risk. The last two columns report the log-likelihood and the average AIC criterion, respectively.

Table 4 reports the estimates of the MIDAS-GAS models. We can see that the Student-t distribution gives a small improvement in terms of fit. Furthermore, including conditional heteroscedasticity improves the in-sample results. Finally, Figure 5 reports the estimated MIDAS weighting functions. Interestingly, we see that estimated functions tend to give more

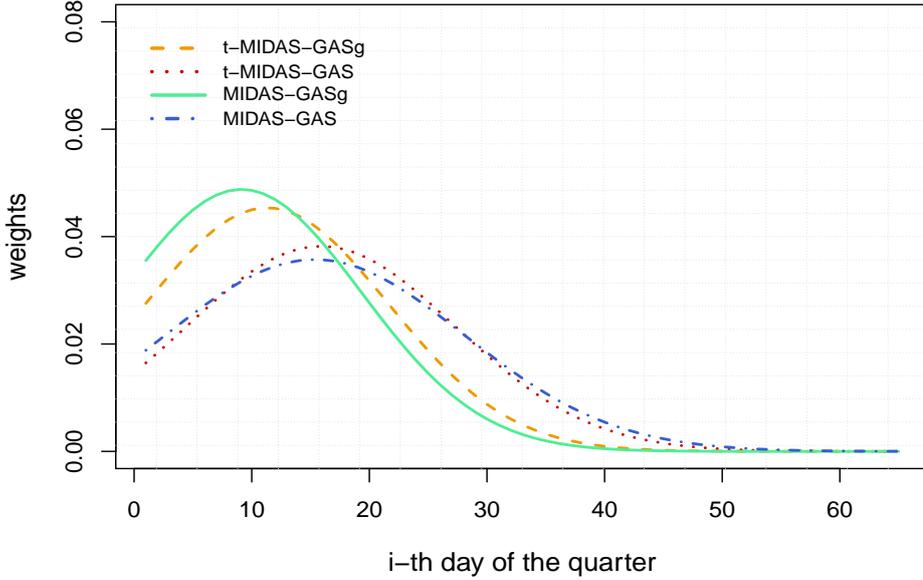


Figure 5: Estimated MIDAS-GAS weighting functions for the four model specifications using GDP growth rate with the measure of financial risk. The horizontal axis indicates the day in the quarter (there are about 63 working days in a quarter).

weight to days in the first and second month of the quarter. This observation is in line with not only the literature reviewed at the beginning of this section, which suggests low volatility followed by increase precedes downward move in output growth, but also with the common wisdom that financial variables are “fast” and macroeconomic variables are “slow” (see e.g. structural vector autoregression analysis on macroeconomic variables and financial risks by [Carriero et al., 2017](#) and [Galí and Gambetti, 2015](#)).

#### 4.2.2 Out-of-sample exercise

We perform a rolling window forecasting exercise as considered in the previous empirical application. The forecasts are obtained from 1 quarter ahead to 6 quarters ahead. The out-of-sample period is from 1989 to 2018. In this case, the time varying mean of MIDAS-GAS models is specified as an autoregressive process of order 2. [Table 5](#) reports the results. We can see that the MIDAS-GAS models tend to have the best performance in terms of point and density forecasts, except for 1 step ahead forecasts where the champion is the multiplicative MIDAS regression extended with Student-t error and conditional heteroscedasticity. Also as is seen easily, the use of a Student-t distribution and the inclusion of conditional heteroscedasticity play an important role in both point and density forecasting. The DM

test shows that several models are significantly out performed by the t-MIDAS-GASg model, especially for density forecasts. Overall, we conclude that MIDAS-GAS models are able to deliver accurate forecasts.

	Forecast Mean Squared Error						Log score criterion					
	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$	$h = 1$	$h = 2$	$h = 3$	$h = 4$	$h = 5$	$h = 6$
t-MIDAS-GASg	1.00	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	<b>1.00</b>	-2.23	<b>-2.22</b>	<b>-2.25</b>	<b>-2.26</b>	<b>-2.29</b>	<b>-2.30</b>
t-MIDAS-GAS	0.99	<b>1.00</b>	1.01	1.01	1.02	1.02	-2.32	-2.31*	-2.33	-2.34	-2.35	-2.35
MIDAS-GASg	1.00	1.01	1.01	1.02	1.02	1.02	-2.23	-2.26	-2.33*	-2.44*	-2.49*	-2.48*
MIDAS-GAS	0.99	1.01	1.01	1.02	1.02	1.02	-2.35*	-2.36**	-2.37*	-2.38	-2.38	-2.39
t-MIDASg(2)	1.01	1.05*	1.04	1.03	1.06	1.08	-2.22	-2.26	-2.33	<b>-2.26</b>	-2.40	-2.43
t-MIDAS(2)	1.00	1.06*	1.06	1.04	1.05	1.07	-2.32	-2.36**	-2.38*	-2.38*	-2.39	-2.41
MIDASg(2)	1.01	1.05*	1.04	1.03	1.06	1.08	-2.26	-2.32*	-2.45	-2.56	-2.66	-2.62
MIDAS(2)	1.01	1.06*	1.05	1.04	1.04	1.07	-2.37*	-2.41***	-2.43**	-2.42**	-2.44*	-2.46**
t-MIDASg(4)	<b>0.98</b>	1.05	1.06	1.07	1.11	1.11	<b>-2.20</b>	-2.29*	-2.37*	-2.38	-2.46*	-2.48*
t-MIDAS(4)	0.99	1.05	1.07*	1.08	1.11	1.11	-2.31	-2.34*	-2.38*	-2.40*	-2.42*	-2.42*
MIDASg(4)	0.99	1.05	1.06	1.08	1.11	1.11	-2.26	-2.37	-2.44	-2.58	-2.69	-2.71
MIDAS(4)	0.99	1.05	1.06	1.08	1.11	1.10	-2.35*	-2.39**	-2.42**	-2.44**	-2.46**	-2.46**
t-ARg(2)	1.00	1.04*	1.05	1.05	1.04	1.04	-2.21	-2.23	-2.28	-2.30	-2.33	-2.35
t-AR(2)	1.00	1.04*	1.04	1.04	1.03	1.03	-2.29	-2.31	-2.34	-2.35	-2.34	-2.35
ARg(2)	1.00	1.05*	1.06	1.06	1.05	1.05	-2.26	-2.31	-2.36	-2.49	-2.53	-2.52
AR(2)	1.00	1.05*	1.05	1.05	1.03	1.03	-2.37*	-2.41***	-2.44***	-2.45**	-2.45*	-2.45*
t-ARg(4)	1.00	1.04*	1.05	1.04	1.04	1.03	-2.21	-2.23	-2.28	-2.30	-2.33	-2.34
t-AR(4)	1.00	1.05	1.05	1.05	1.03	1.03	-2.28	-2.30	-2.34	-2.34	-2.34	-2.34
ARg(4)	1.00	1.05*	1.06	1.05	1.05	1.05	-2.26	-2.31	-2.36	-2.47	-2.53	-2.52
AR(4)	0.99	1.04	1.05	1.04	1.03	1.02	-2.37*	-2.39**	-2.41**	-2.41**	-2.41*	-2.41*
t-GASg	1.02	1.06**	1.05*	1.04	1.02	1.01	-2.23	-2.24	-2.28	-2.30	-2.32	-2.33
t-GAS	1.02	1.05**	1.05*	1.03	1.02	1.01	-2.32	-2.33*	-2.35*	-2.34	-2.34	-2.34
GASg	1.02	1.05*	1.05*	1.04	1.02	1.01	-2.29	-2.30*	-2.33*	-2.37	-2.37	-2.38
GAS	1.00	1.05**	1.05*	1.04	1.02	1.01	-2.37*	-2.39**	-2.41**	-2.41*	-2.41*	-2.40
fMIDAS	1.02	1.07**	1.06*	1.06	1.04	1.04	-2.37*	-2.41***	-2.44***	-2.45**	-2.45*	-2.45*

Table 5: Relative MSE and log score criterion for the different model specifications using quarterly GDP growth rate with the daily measure of financial risk. The first 6 columns of the table report the ratio between the MSE of each model with respect to the MSE of benchmark model (t-MIDAS-GASg) for several forecasting horizons (from 1 to 6 steps ahead). A value greater than 1 indicates that a model is underperforming the benchmark model instead the opposite is true when the MSE ratio is smaller than 1. The last 6 columns report the log score criterion for several forecasting horizons (from 1 to 6 steps ahead). Stars indicate the significance level of the DM test (\*\*\*0.1%, \*\*1% and \*5%). Bold characters indicate the best performing model for a given horizon.

### 4.2.3 Nowcasting

In this section, we employ the MIDAS-GAS approach to nowcast GDP. Nowcasting can be implemented by simply shifting the time index of the high frequency variable. Assume that the aim is to predict  $y_{T+1}$  and the last observations available are  $y_T$  and  $x_{s,T+1}$ . For instance, in the GDP application, this means that in quarter  $T + 1$  the financial risk measure is observed up to day  $s$ . In this nowcasting setting, we define the  $s$ -periods shifted variable

$x_t^s$  as  $x_t^s = (x_{1+s,t}, \dots, x_{n_x,t}, x_{1,t+1}, \dots, x_{s,t+1})'$ . The specification of the MIDAS-GAS filter becomes

$$f_{t+1} = \delta + \beta f_t + \alpha_y s_t^y + \alpha_x \sum_{i=1}^{n_x} \omega_i(\varphi) s_{i,t}^{x,s},$$

where  $s_{i,t}^{x,s}$  denotes the score innovation from the shifted variable  $x_{i,t}^s$ , which is the  $i$ -th element of the vector  $x_t^s$ . The estimation of the model is performed by WML as discussed before.

	One month		Two months	
	FMSE	Log score	FMSE	Log score
t-MIDAS-GASg	1.02	-2.20	1.03	-2.20
t-MIDAS-GAS	1.01	-2.30*	1.00	-2.30*
MIDAS-GASg	1.00	-2.20	1.00	-2.20
MIDAS-GAS	1.00	-2.35**	<b>0.99</b>	-2.34**
t-MIDASg(2)	1.01	-2.19	1.02*	<b>-2.19</b>
t-MIDAS(2)	1.02*	-2.30*	1.02	-2.30*
MIDASg(2)	1.02*	-2.23	1.02*	-2.23
MIDAS(2)	1.01	-2.35**	1.01	-2.35**
t-MIDASg(4)	<b>0.99</b>	<b>-2.18</b>	1.00	<b>-2.19</b>
t-MIDAS(4)	1.00	-2.30	1.00	-2.29
MIDASg(4)	1.00	-2.23	1.00	-2.24
MIDAS(4)	0.99	-2.34*	1.02	-2.35**
fMIDAS	1.06*	-2.42***	1.10	-2.42***

Table 6: Relative MSE and log score criterion for the different model specifications using quarterly GDP growth with daily financial risk measure. The first 2 columns of the table report the relative MSE with respect to benchmark model (MIDAS-GASg) and the log score criterion for 1 month ahead nowcasting. The last 2 columns report the results for 2 months ahead nowcasting. Stars indicate the significance level of the DM test (\*\*0.1%, \*1% and \*\*5%). Bold characters indicate the best performing model for a given horizon.

We consider nowcasting of quarterly GDP growth utilizing the daily risk measure of financial condition that becomes available for the first one and two months of the corresponding quarter. For example, in the one month nowcasting, the vector of daily measures that enters the models includes observations in the last two months of the previous quarter plus those in the first month of the most recent quarter. Table 6 reports the results of the nowcasting exercise. The results show a similar performance (our benchmark model here is MIDAS-GASg) across the models except for density predictions, where models without conditional heteroscedasticity have a significantly lower accuracy compared to MIDAS-GASg. Overall, we can conclude that the MIDAS-GAS models and multiplicative MIDAS regressions

perform comparably well in this nowcasting exercise.

## 5 Conclusion

In this paper, we have introduced a novel approach for forecasting and nowcasting with mixed frequency data: the MIDAS-GAS model. The MIDAS-GAS model transforms the score contributions of the high frequency variables through a MIDAS weighting scheme. The proposed approach has several advantages as it retains all the appealing features of GAS models while accounting for mixed frequencies. Based on the general MIDAS-GAS framework, we have developed a novel forecasting model with dynamic factor structures for location and scale. The method has shown a promising forecasting performance in two empirical applications on forecasting U.S. headline inflation using crude oil prices and forecasting GDP growth using a measure of financial condition.

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